Q1

(i)

Let $A = \pi/2$ and let $B = \pi/4$. Then

$\sin(A-B) = \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin A + \sin B = \sin\frac{\pi}{2} + \sin\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2} \neq 1 + \frac{\sqrt{2}}{2}$

Therefore $\sin(A-B) = \sin A + \sin B$ is not true in general.

(ii)

Let $A = \pi$ and let $B = \pi/3$. Then

$\sin(A-B) = \sin(\pi - \pi/3) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

and $\sin A + \sin B = \sin\pi + \sin\frac{\pi}{3} = 0 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

With reference to part (ii), note the compound angle formula

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

If $\cos A = -1$, then $\sin A = 0$ and

$\sin(A-B) \equiv (0)\cos B - (-1)\sin B = 0 + \sin B = \sin A + \sin B$

[this is independent of B!]

If $\cos B = 1$, then $\sin B = 0$ and

$\sin(A-B) \equiv \sin A(1) - \cos A(0) = \sin A + 0 = \sin A + \sin B$

[this is independent of A!]

So any $A = (2k+1)\pi$ and/or $B = 2k\pi$, where k is an integer, will make the equation true

Q2A

a)

$$\sin(X+Y-Z) = \sin((X+Y)-Z)$$

$$\equiv \sin(X+Y)\cos Z - \cos(X+Y)\sin Z$$

$$\equiv (\sin X\cos Y + \cos X\sin Y)\cos Z - (\cos X\cos Y - \sin X\sin Y)\sin Z$$

$$= \sin X\cos Y\cos Z + \cos X\sin Y\cos Z - \cos X\cos Y\sin Z + \sin X\sin Y\sin Z$$

Therefore

$$\sin(X+Y-Z) \equiv \sin X\cos Y\cos Z + \cos X\sin Y\cos Z - \cos X\cos Y\sin Z + \sin X\sin Y\sin Z$$

Q2B

b)

Let $X = 180$, $Y = 30$, $Z = 45$. Then

$$X+Y-Z = 180+30-45 = 165$$

so

$$\sin(165^\circ) = \sin(180+30-45)$$

$$= \sin(180)\cos(30)\cos(45) + \cos(180)\sin(30)\cos(45) - \cos(180)\cos(30)\sin(45) + \sin(180)\sin(30)\sin(45)$$

$$= (0)(\sqrt{3}/2)(\sqrt{2}/2) + (-1)(1/2)(\sqrt{2}/2) - (-1)(\sqrt{3}/2)(\sqrt{2}/2) + (0)(1/2)(\sqrt{2}/2)$$

$$= 0 - \sqrt{2}/4 + \sqrt{6}/4 + 0$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Q3

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin 2A \equiv 2\sin A\cos A$$ [Double angle formula]

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$ [Double angle formula]

Because $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ for any θ,

$$\tan 2A \equiv \frac{\sin 2A}{\cos 2A}$$

Using the double angle formulae for sin and cos

$$\frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Dividing the numerator and denominator by $\cos^2 A$ gives

$$\frac{2\sin A/\cos A}{1 - \frac{\sin^2 A}{\cos^2 A}}$$

But $\frac{\sin A}{\cos A} \equiv \tan A$, therefore

$$\frac{2\sin A/\cos A}{1 - \frac{\sin^2 A}{\cos^2 A}} \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Combining the above,

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Q4

Given that $a\sin\theta + b\cos\theta$, where a and b are positive constants, is to be written in the form $R\sin(\theta+\alpha)$, find expressions for:

(i) α in terms of a and b

(ii) R in terms of a and b

[6]

$\sin(A+B) \equiv \sin A\cos B + \cos A \sin B$ [Compound angle formula]

$\tan A \equiv \dfrac{\sin A}{\cos A}$

$\sin^2 A + \cos^2 A \equiv 1$

Note: It's important to show in part (ii) why R is the positive square root of a^2+b^2!

$R\sin(\theta+\alpha) \equiv R\cos\alpha\sin\theta + R\sin\alpha\cos\theta = a\sin\theta + b\cos\theta$

if $R\cos\alpha = a$ and $R\sin\alpha = b$

(i) $\dfrac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{b}{a}$

$\Rightarrow \boxed{\alpha = \arctan\left(\frac{b}{a}\right)}$

(ii) $(R\cos\alpha)^2 + (R\sin\alpha)^2 = a^2+b^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = R^2(\cos^2\alpha + \sin^2\alpha) = R^2 = a^2+b^2$

$R = \pm\sqrt{a^2+b^2}$

But, because $a>0$ and $b>0$,

$\frac{b}{a} > 0 \Rightarrow 0 < \arctan\left(\frac{b}{a}\right) < \frac{\pi}{2} \Rightarrow 0 < \alpha < \frac{\pi}{2}$

$\Rightarrow \sin\alpha > 0$ and $\cos\alpha > 0 \Rightarrow R > 0$

Therefore

$\boxed{R = \sqrt{a^2+b^2}}$

Q5A

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

a) Use the double angle formula for cos to write

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$

Find principal values for θ, then use symmetry properties of cos to find other solutions in the interval:

If $\cos\theta = 1$, $\theta = \cos^{-1}(1) = 0$

If $\cos\theta = -\frac{1}{2}$, $\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

or $\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

$\boxed{\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$

Q5B

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A} \quad \begin{bmatrix}\text{Double angle} \\ \text{formula}\end{bmatrix}$$

b) Use the double angle formula for tan to write

$$\frac{2\tan x}{1-\tan^2 x} = 3\tan x \Rightarrow 2\tan x = 3\tan x - 3\tan^3 x$$

$$\Rightarrow 3\tan^3 x - \tan x = 0 \Rightarrow \tan x(3\tan^2 x - 1) = 0$$

So $\tan x = 0$ or $\tan x = \pm\frac{1}{\sqrt{3}}$

Find principal values for x, then use symmetry properties of tan to find other solutions in the interval:

If $\tan x = 0$, $x = \tan^{-1}(0) = 0$

or $x = -\pi$ or $x = \pi$

If $\tan x = \frac{1}{\sqrt{3}}$, $x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

or $x = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

If $\tan x = -\frac{1}{\sqrt{3}}$, $x = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

or $x = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

$$\boxed{x = -\pi, -\frac{5\pi}{6}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi}$$

Q6

Show that

$$\tan 2\theta \tan\theta \equiv \sec 2\theta - 1$$

[5]

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A} \quad \begin{bmatrix}\text{Double angle} \\ \text{formula}\end{bmatrix}$$

$$\tan A \equiv \frac{\sin A}{\cos A}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A \quad \begin{bmatrix}\text{Double angle} \\ \text{formula}\end{bmatrix}$$

$$\tan 2\theta \tan\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}(\tan\theta) = \frac{2\tan^2\theta}{1-\tan^2\theta}$$

$$\equiv \frac{2\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{1-\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\frac{2\sin^2\theta}{\cancel{\cos^2\theta}}}{\frac{\cos^2\theta - \sin^2\theta}{\cancel{\cos^2\theta}}}$$

$$= \frac{2\sin^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$\equiv \frac{1-\cos 2\theta}{\cos 2\theta}$$

$$= \frac{1}{\cos 2\theta} - 1 \equiv \sec 2\theta - 1$$

Therefore

$$\tan 2\theta \tan\theta \equiv \sec 2\theta - 1$$

Q7a

a)

$R\sin(\theta-\alpha) \equiv R\cos\alpha\sin\theta - R\sin\alpha\cos\theta = 5\sin\theta - 3\cos\theta$

<u>if</u> $R\cos\alpha = 5$ and $R\sin\alpha = 3$

$\sin^2\alpha + \cos^2\alpha \equiv 1$

$(R\cos\alpha)^2 + (R\sin\alpha)^2 = R^2(\cos^2\alpha + \sin^2\alpha) = R^2 = 5^2 + 3^2 = 25 + 9 = 34$

So $R^2 = 34 \implies R = \sqrt{34}$

$\frac{\sin\alpha}{\cos\alpha} \equiv \tan\alpha$

$\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{3}{5}$

So $\tan\alpha = \frac{3}{5} \implies \alpha = \tan^{-1}\left(\frac{3}{5}\right) = 0.540$ (3 s.f.)

Q7b

(a) Show that $5\sin\theta - 3\cos\theta$ can be written in the form $R\sin(\theta-\alpha)$ where $R = \sqrt{34}$, and $\alpha = 0.540$ radians correct to three significant figures.

[4]

(b) Use your result from part (a), and the properties of the sine and cosine functions, to solve the equation

$$3\cos 2x + 5\sin 2x = 0.4 \qquad 0 \le x \le 2\pi$$

[5]

$\cos(-A) = \cos A \qquad \sin(-A) = -\sin A$

[from symmetry properties of cos and sin]

b) $3\cos 2x + 5\sin 2x = 3\cos(-2x) - 5\sin(-2x)$

$= -(5\sin(-2x) - 3\cos(-2x)) = -\sqrt{34}\sin(-2x-\alpha)$ ← $\theta = -2x$

So $-\sqrt{34}\sin(-2x-\alpha) = 0.4$

$\sin(-2x-\alpha) = -\frac{0.4}{\sqrt{34}} \implies \sin(2x+\alpha) = \frac{0.4}{\sqrt{34}}$

$0 \le x \le 2\pi \implies \underset{\approx 0.540}{\alpha} \le 2x+\alpha \le \underset{\approx 13.1}{4\pi+\alpha}$ } Transform the solution interval

Solve for $2x+\alpha$ in the transformed interval:

$2x+\alpha = \sin^{-1}\left(\frac{0.4}{\sqrt{34}}\right) = 0.068653\ldots = q$ } outside the interval

or $2x+\alpha = \pi - q = 3.072939\ldots$

or $2x+\alpha = q + 2\pi = 6.351838\ldots$

or $2x+\alpha = (\pi - q) + 2\pi = 9.356124\ldots$

or $2x+\alpha = q + 4\pi = 12.635023\ldots$

} by symmetry properties of sin

AT VERY end, convert to solutions for x:

Subtracting $\alpha = \tan^{-1}(3/5)$ from those solutions and dividing by 2 gives

$\boxed{x = 1.27,\ 2.91,\ 4.41,\ 6.05 \text{ (3 s.f.)}}$

Q8a

s 4*A*, in terms of cos *A*.

[4]

[5]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Dou... f...

$\cos 4A = \cos(2(2A))$

a) $\cos 4A \equiv 2\cos^2 2A - 1$

$\equiv 2(2\cos^2 A - 1)^2 - 1$

$= 2(4\cos^4 A - 4\cos^2 A + 1) - 1$

$= 8\cos^4 A - 8\cos^2 A + 2 - 1$

$= 8\cos^4 A - 8\cos^2 A + 1$

So

$$\boxed{\cos 4A \equiv 8\cos^4 A - 8\cos^2 A + 1}$$

Q8b

(a) Use an identity for cos 2*A* to derive an identity for cos 4*A*, in terms of cos *A*.

[4]

(b) Hence, or otherwise, solve the equation

$2\cos 4x = 7\sin^2 x - 2 \qquad 0 \le x \le \pi$

[5]

From part (a),

$\cos 4A \equiv 8\cos^4 A - 8\cos^2 A + 1$

$\sin^2 x + \cos^2 x \equiv 1$

b) $2(8\cos^4 x - 8\cos^2 x + 1) = 7(1 - \cos^2 x) - 2$

$16\cos^4 x - 9\cos^2 x - 3 = 0$ } this is a quadratic in $\cos^2 x$

$\cos^2 x = \dfrac{9 \pm \sqrt{273}}{32}$ } quadratic formula

But $\cos^2 x \geq 0$, so $\cos^2 x = \dfrac{9 + \sqrt{273}}{32}$ } reject negative answer

$\Rightarrow \cos x = \pm\sqrt{\dfrac{9+\sqrt{273}}{32}}$

So

$x = \cos^{-1}\left(\sqrt{\dfrac{9+\sqrt{273}}{32}}\right) = 0.466659\ldots$

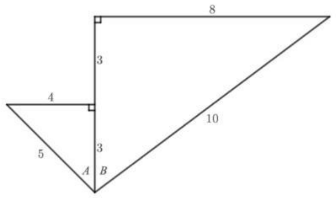
or $x = \cos^{-1}\left(-\sqrt{\dfrac{9+\sqrt{273}}{32}}\right) = 2.674932\ldots$

Rounding to 3 s.f. gives

$$\boxed{x = 0.467,\ 2.67 \ (3 \text{ s.f.})}$$

Q9

The diagram below shows two right-angled triangles.
Angles A and B have been labelled.



Given that $\alpha = A + B$, find the exact values of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

[7]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\sin^2 A + \cos^2 A \equiv 1$

$\tan A \equiv \dfrac{\sin A}{\cos A}$

A and B are angles in right-angled triangles, therefore A and B are acute.

Also $\sin A = \frac{4}{5} = \frac{8}{10} = \sin B$

Therefore $A = B$ and $\alpha = A + B = A + A = 2A$

So $\sin\alpha = \sin 2A = 2\sin A\cos A = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$

Then

$\cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}$

Note that

$\sin A = \frac{4}{5} > \frac{\sqrt{2}}{2} = \sin 45°$

$\Longrightarrow A > 45° \Longrightarrow 90° < 2A < 180°$

$\Longrightarrow \alpha$ is obtuse

So $\cos\alpha = -\frac{7}{25}$

Then $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{24/25}{-7/25} = -\dfrac{24}{7}$

$$\boxed{\sin\alpha = \frac{24}{25} \quad \cos\alpha = -\frac{7}{25} \quad \tan\alpha = -\frac{24}{7}}$$

Q10

(i) Explain briefly why $\theta = 0$ is **not** a solution to the equation $3\theta\cot 2\theta = 0$.

(ii) By using an appropriate approximation, determine the value of

$$\lim_{\theta\to 0} 3\theta\cot 2\theta$$

[4]

$\cot A \equiv \dfrac{\cos A}{\sin A}$

Small angle approximations

If θ is 'small' (close to zero) and measured in radians, then

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$\tan\theta \approx \theta$

(i) $3\theta\cot 2\theta = 3\theta\dfrac{\cos 2\theta}{\sin 2\theta} = \dfrac{3\theta\cos 2\theta}{\sin 2\theta}$

when $\theta = 0$, $3\theta\cot 2\theta = \dfrac{3(0)\cos(0)}{\sin(0)} = \dfrac{0}{0}$

which is undefined

(ii) $\theta \to 0$ means θ is 'small', so we can use the small angle approximations.

Noting that $\cot 2\theta = \frac{1}{\tan 2\theta}$, we have

$$\lim_{\theta\to 0} 3\theta\cot 2\theta = \lim_{\theta\to 0}\frac{3\theta}{\tan 2\theta}$$

$$= \lim_{\theta\to 0}\frac{3\theta}{2\theta}$$

$$= \lim_{\theta\to 0}\frac{3}{2}$$

$$= \frac{3}{2}$$

Therefore

$$\boxed{\lim_{\theta\to 0} 3\theta\cot 2\theta = \frac{3}{2}}$$

Q11a

a) $a\sin x + b\cos x = R\sin(x+\alpha)$

$a = R\cos\alpha \quad b = R\sin\alpha$

$R = \sqrt{a^2+b^2}$

$R = \sqrt{115^2 + (115\sqrt{3})^2} = \sqrt{52900}$

$R = 230$

$R\cos\alpha = 115 \quad R\sin\alpha = 115\sqrt{3}$

$\tan\alpha = \dfrac{115\sqrt{3}}{115} = \sqrt{3}$

$\alpha = \dfrac{\pi}{3}$

$$\boxed{115\sin\omega t + 115\sqrt{3}\cos\omega t = 230\sin\left(\omega t + \frac{\pi}{3}\right)}$$

Q11b

b) $\omega = 2\pi f = 2\pi \times 50 = 100\pi$

i) $V = 230 \sin\left(\omega t + \frac{\pi}{3}\right)$

$t = 0$ $\quad V = 230 \sin\left(\frac{\pi}{3}\right) = 199.185\ldots$

$$\boxed{V = 199 \text{ VOLTS (3SF)}}$$

ii) $V < 0$

$230 \sin\left(100\pi t + \frac{\pi}{3}\right) = 0$

$100\pi t + \frac{\pi}{3} = 0$ CANNOT BE NEGATIVE

$100\cancel{\pi} t + \frac{\cancel{\pi}}{3} = \cancel{\pi}$ ADJUST BY π

$$\boxed{t = \frac{1}{150} \text{ SECONDS}}$$

Q11c

c)

i) $\frac{100\pi}{2\pi} = 50$ $\quad \boxed{50 \text{ SECONDS}}$

ii) $\frac{1}{60}$ SECONDS $\quad \omega \times \frac{1}{60} = 2\pi$

$\omega = 120\pi$

$$f = \frac{\omega}{2\pi} = \frac{120\cancel{\pi}}{2\cancel{\pi}} = \boxed{60 \text{ Hz}}$$